A CLASS OF CONGRUENCIES ON DISTRIBUTIVE SEMILATTICE

UNA CLASE DE CONGRUENCIAS EN SEMIRRETÍCULO DISTRIBUTIVO

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ABSTRACT

In this paper we, contribute the notation of natural epimorphism of a semilattice on the quotient semilattice and subsemilattice. If S is distributive semilattice and F is a filter of S, then we demonstrate that θ_F is the smallest congruence on S containing F in a single equivalence class and that S/θ_F is distributive. In addition, the author proved that map $F \longmapsto \theta_F$ is an isomorphism from the lattice of $F_0(S)$ all non-empty filters of S into a permutable sublattice of the lattice $C(S)$ of all congruencies on S.

Keywords: Congruence class of semilattice; Distributive Semilattice; Natural epimorphism of semilattice; Quotient semilattice.

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RESUMEN

En este trabajo contribuimos con la notación del epimorfismo natural de una semirredura sobre el cociente semirreticulado y subsemretículo. Si S es una semirrejilla distributiva y F es un filtro de S, entonces demostramos que θ_F es la congruencia más pequeña en S que contiene F en una sola clase de equivalencia y que S/θ_F es distributiva. Además, el autor demostró que el mapa $F \longmapsto \theta_F$ es un isomorfismo de la red $F_0(S)$ de todos los filtros no vacíos de S en una subred permutable de la red $C(S)$ de todas las congruencias en S.

Palabras clave: clase de congruencia de semirredura; Semirretículo distributivo; Epimorfismo natural de semirreduras; Cociente de semirreduras.

INTRODUCTION

In [14] (Klein Barmen 1939) introduced the concept of semilattice [13]. A semilattice is a partially ordered set in which any two elements have a greatest lower bound, but not necessarily upper bound. The class of distributive semilattices is a significant subclass of semilattices. Then, several authors were studied the class of distributive semilattices. The author refer the reader to [1], [2], [3], [4], [7], [9], [11], [17], [18], [19], and [21] for distributive semilattices. The concept of 0-distributive semilattices is another important extension of the class of distributive semilattices. In [19] (J.C.Varlet 1968) first introduced the concept of 0-distributive lattices. Then many authors including [3], [6], [10], [15], [16], and [20] studied concept of 0-distributive for lattices and semilattices.

Let S be a semilattice. A non-empty subset L of S is called directed above if, for any $a,b \in L$ there exists $c \in L$ such that $a \leq c$ and $b \leq c$. L is said to be a final segment if, for any a L and $x \in S$, $a \leq x$ implies $x \in L$. In addition, a non-empty subset L is called directed below if, for any, $a,b \in L,$ there exists $c \in L$ such that $c \leq a$ and $c \leq b$. L is called an initial segment if, for any $a \in I$ and $x \in S,$ $x \leq a$ implies $x \in I$. Following this concept, many authors developed the concept of filters and ideals of semilattice. The we refer the readers to [7], [16], [17] , and [18].

Let S and L be semilattices. A map $f : S \to L$ is said to be a homomorphism, if f is join preserving and meet preserving, that is, for all $a,b \in S$.

$$f(a \vee b)= f(a)\vee f(b) and\ f(a \wedge b)= f(a) \wedge f(b)$$

A bijective homomorphism is a semilattice isomorphism [22]. If $f : L \to K$ is a one-to-one homomorphism, then the sub-semilattice of $f(S\}$ L is isomorphic to S and we refer to f as an embedding (of S into L). In [18] (Swamy 1979) introduced the class of natural epimorphism of a semilattice on the quotient semilattice, and he denoted the natural epimorphism of semilattice S on the quotient semilatticeby S/θ_F by π_F. For any sub-semilattice L of a semilattice S, define

$$\theta_L =\{(x, y)\in SxS :\ x\wedge a = y\wedge a\ for\ some\ a \in L\}$$

After (Swamy 1979) nobody gives attention to the concept of congruence classes on the distributive semilattice as far as we investigate it. Therefore, by following this, we consider and demonstrate that; map $F \longmapsto \theta_F$ is an isomorphism from the lattice $F_0(S)$ of the class of all non-empty filters of S into a permutable sublattice of the lattice $C(S)$ of the classes of all congruencies on S. In addition, we prove that

θ_F is the smallest congruence on S containing F in a single equivalence class and that S/θ_L is distributive.

The manuscript is structured as follows. In this section, we introduced the background of semilattice with basic results about semilattices which are already exists and needed in developing this manuscript. In Section 2, we introduced the concepts of distributive semilattices. Some important Theorems and Corollaries in distributive semilattices and pseudo-complemented semilattice were also presented in this section. In section 3, we will prove the theorems belonging to classes of congruence in distributive semilattice.

PRELIMINARIES

In this section, the author will present some necessary notations and definitions.

Definition: 2.1. [2] Let S be a semilattice. A non-empty subset F of S is called a filter if;

(i) $a, b \in F$ implies $a \wedge b \in F$ and

(ii) $a \in F$, $b \in S$ with $a \leq b$ implies $b \in F$.

Definition: **2.2.** [3] A non-empty subset I of S is called an ideal if;

(i) For any $a, b \in I$, there exist $c \in I$ such that $a \leq c, b \leq c$ and

(ii) $a \in I, b \in S$ With $b \leq a$ implies $b \in I$.

A filter (ideal) F of a semilattice S is called a proper filter (ideal) if $\neq S$. A maximal filter (ideal) F of S is a proper filter (ideal), which is not contained in any other proper filter (ideal), that is, if there is a proper filter (ideal) G such that $F \subseteq G$, then $F = G$. (see [4])

Let $F(S)$ and $I(S)$ denotes the set of all filters of S and the set of all ideals of S respectively. Then, for any $a \in S$, let $(a]$ denote the ideal $\{x \in S : x \leq a\}$ and $[a)$ denotes the filter $\{x \in S : a \leq x\}$. (see[11])

Definition: 2.3. [5] A semilattice S is said to be distributive if, for any $a,b,c \in S$, such that $a \wedge b \leq c$, there exist $x, y \in S$ such that $a \leq x$, $b \leq y$, and $x \wedge y = c$.

Lemma 2.4. [4] If S is a distributive semilattice, S is directed above.

Proof. Suppose S be a distributive semilattice, then for any $a,b \in S$, $a \wedge b \in S$ and $a \wedge b \leq b$. There exists $x,y \in S$ such that $a \leq x$, $b \leq y$, and $x \wedge y = b$. Also $b = x \wedge y \leq x$. Therefore, for any $a,b \in S$, there exists $x \in S$ such that $a \leq x$ and $b \leq x$.

Hence S is directed above.

Theorem 2.5. [6] Let I be an ideal of S and J a filter of S such that $I \cap J \neq \emptyset$. Then, $I \cap J$ is a distributive sub-semilattice of S.

Proof. Suppose I be an ideal of S and J a filter of S such that $I \cap J \neq \emptyset$. Let $a,b \in I \cap J$, then $a \wedge b \in I$. Since I is the initial segment, there exist c $J \in$ such that $c \leq a$ and $b \leq c$.

This implies $c \leq a \wedge b$ and $a \wedge b \in J$ since J is the final segment. Therefore $I \cap J$ is a sub-semilattice of S.

Let $a,b,c \in I \cap J$ such that $a \wedge b \leq c$. There exists $x,y \in S$ such that $a \leq x$, $b \leq y$, and $x \wedge y = c$. Since I is an ideal of S, there exist $z \in I$ such that $a \leq z$, $b \leq z$ and $c \leq z$.

Now $c = c \wedge z = (x \wedge z) \wedge (y \wedge z)$, $a \leq x \wedge z \in I \cap J$ and $b \leq y \wedge z \in I \cap J$. Therefore $I \cap J$ is distributive sub-semilattice.

Definition 2.6. [19] A semilattice S with 0 is called 0 - distributive if for any $a,b,c \in S$ such that $a \wedge b = 0 = a \wedge c$ implies $a \wedge d = 0$ for some $d \geq b,c$.

Proposition 2.7. [16] Let S be distributive semilattice. Let I be an ideal of S and J a filter of S such that $I \cap J \neq \emptyset$ then, $I \cap J$ is a distributive sub-semilattice of S.

Proof: Suppose that I is an ideal of S and J is a filter of S such that $I \cap J \neq \emptyset$. Let $a,b \in I \cap J$, then $a \wedge b \in I$. Since I is the initial segment, there exist $c \in J$ such that $c \leq a$ and $b \leq c$. Hence $c \leq a \wedge b$ and $a \wedge b \in J$, since J is the final segment. Thus $I \cap J$ is a sub-semilattice of S. Let $a,b,$ and $c \in I \cap J$ such that $a \wedge b \leq c$. There exists x and $y \in S$ such that $a \leq x$, $b \leq y$ and $x \wedge y = c$. Since I is an ideal of S, there exist $z \in I$ such that $a \leq z$, $b \leq z$, and $c \leq z$. Then we have $c = c \wedge z = (x \wedge z) \wedge (y \wedge z)$, $a \leq x \wedge z \in I \cap J$ and $b \leq y \wedge z \in I \cap J$. Hence $I \cap J$ is distributive.

Theorem 2.8. [18] Let $(S, \wedge, \vee)$ be a lattice. Then the following are equivalent:

1. $(S, \wedge, \vee)$ is a distributive lattice.
2. $(S, \wedge)$ is a distributive meet semilattice.
3. $(S, \vee)$ is a distributive join semilattice.

Corollary 2.9. [18] Every non-empty ideal (filter) of S is a distributive sub-semilattice of S.

Proof: Suppose that I is a non-empty ideal of S. Let $a,b \in I$ such that $a \wedge b \leq c$. There exist $x,y \in S$ such that $a \leq x$ and $b \leq y$, and $c = c \wedge y$ as S is distributive. There exist $z \in I$ such that $a \leq z$, $b \leq z$ and $c \leq z$ as I is an ideal of S.

Now. $c = c \wedge z = (x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z) \in I$

Hence I is distributive.

Similarly, suppose that J is a nonempty filter of S. For any $a,b \in J$, there exist $c \in J$ such that $c \leq a$ and $c \leq b \Rightarrow c \leq a \wedge b \Rightarrow a \wedge b \in J$ as J is the final segment. Then, J is a sub-semilattice of S. Let $a,b,c \in J$ such that $a \wedge b \leq c$, there exist $x,y \in S$ such that $a \leq x$, $b \leq y$, and $x \wedge y = c$.

Now as a,b and $c \in J$, then $x,y \in J$. Hence $x \wedge y = c \in J$.

Therefore J is distributive.

Theorem 2.9. Let I be an ideal of S and L is a sub-semilattice of S such that $I \cap L = \emptyset$, then, there exists a prime ideal P of S such that $I \subset P$ and $P \cap L = \emptyset$.

Proof: Suppose that $T = \{x \in S : a \leq x$ for some $a \in L\}$. Then T is the filter of S and $T \cap I = \emptyset$ and by Zorn's lemma, there exists a filter F of S which is maximal among all filters containing L, and disjoint from I. Now $I \subset S - F$ and $(S-F) \cap L = \emptyset$.

Let us prove F is a prime filter. Let $x,y \in S- F$, then by maximalist of F there exists

$$a \in I \cap (F \vee [x)) \text{ and } b \in I \cap (F \vee [y))$$

There exist $c \in I$ such that $a \leq c$ and $b \leq c$

$\Rightarrow c \in I \cap (F \vee [x)) \cap (F \vee [x)) = I \cap (F \vee ([x) \cap [y)) \Rightarrow [x) \cap [y) \not\subset F$

Therefore F is a prime filter of S.

Theorem 2.10. [21] Every maximal ideal (filter) of S is prime.

Proof: Let S be distributive semilattice and Q is a maximal filter of S. Then S-Q is minimal ideal. Then there exist a prime ideal I such that S-$Q \subseteq I$ then $Q \cap I = \emptyset$.

Let $x,y \notin Q$, then $x,y \in S$-Q. Then there exist $a \in I \cap (Q \wedge [x))$ and $b \in I \cap (Q \wedge [y))$.

This implies there exist $c \in I$ such that $a \leq c$ and $b \leq c$, since I directed above.

Hence
$c \in I \cap (Q \vee [x)) \cap (Q \vee [y))) = I \cap (Q \vee ([x) \cap [y)) \Rightarrow [x) \cap [y) \nsubseteq Q$

Therefore Q is a prime filter of S. Hence the result.

Theorem 2.11. [17] Let S be a semilattice. S has a largest element if and only if the intersection of all nonempty filters is again non-empty.

Proof: Suppose that S has, a largest element says t, then $t \in \bigcap_{F \in F(S)} F$. Conversely suppose that the intersection of all nonempty filters (ideals) is again non-empty, by proposition 2.3, S has a Largest (smallest) element.

Lemma 2.12. [17] Let $P \subset S$. Then P is a prime filter of S if and only if S - P is a prime ideal.

Proof: Suppose P is a prime filter of S, it is non-empty and proper. To show S - P is directed above, let $a,b \notin S$ - P. Suppose if possible $[a) \cap [b) \cap S$-$P = \emptyset$, then $[a) \cap [b) \subseteq P$. This implies $[a) \subseteq P$ or $[b) \subseteq P$. Since P is prime then $a \notin S$ - P or $b \notin S$ - P. This is a contradiction. Therefore $([a) \cap [b)) \cap S$-$P \neq \emptyset$.

So S - P is an ideal. To show that S - P is prime, suppose
$a \wedge b \in S - P \Rightarrow a \wedge b \notin P \Rightarrow \ a \notin P$ or $b \in P$

Hence $a \in S$ - P or $b \in S$ - P.

Conversely, suppose S - P be a prime ideal, then S - $P \neq \emptyset$, S. So that $P \neq \emptyset$, S. Since S - P is directed above, and then P is directed below. If $a,b \in P$ then $a,b \notin S$ - P. Since S - P is prime, $a \wedge b \notin S$ - P. Thus $a \wedge b \in P$, and so P is a filter.

Now, to show that P is a prime filter, we suppose if possible $Q \cap R \subseteq P$ and $Q \nsubseteq P,\ R \nsubseteq P \Rightarrow Q \cap S - P \neq \emptyset$ and $R \cap S$-$P \neq \emptyset$, then there exists $a \in Q \cap S$ - P and $b \in R \cap S - P \Rightarrow \exists c \in S - P$ with $a, b \leq c \Rightarrow c \in P \cap S - P$. This is a contradiction.

Therefore $Q \subseteq P$ *or* $R \subseteq P \Rightarrow P$ is prime.

CONGRUENCIES ON DISTRIBUTIVE SEMILATTICE

Definition 3.1. Let S be a semilattice. A homomorphism between two meet semilattices S and T is a map $f : S \to T$ with the property that $f(a \wedge b) = f(a) \wedge f(b)$, where $a,b \in S$. A semilattice homomorphism is called semilattice isomorphism

Theorem 3.2. Let S be any semilattice, which is directed above. Then S is distributive if and only if S/θ_L is distributive for every subsemilattice L of S.

Proof: Suppose S be a distributive semilattice and L a subsemilattice of S. Let $x,y,z \in S$ such that
$\pi_L(x) \wedge \pi_L(y) \leq \pi_L(z) \Rightarrow \exists a \in L$ such that $x \wedge y \wedge a = x \wedge y \wedge z \wedge a. \Rightarrow (x \wedge a) \wedge (y \wedge a) \leq z \Rightarrow \exists\, b, c \in S$ such that $x \wedge a \leq b$, $y \wedge a \leq c$ and $b \wedge c = z$

Now, $\pi_L(x \wedge a) \leq \pi_L(b)$, $\pi_L(y \wedge a) \leq \pi_L(c)$ and $\pi_L(b) \wedge \pi_L(c) = \pi_L(z)$. Therefore S/θ_L is distributive for every subsemilattice L of S.

Conversely, let $a,b,$and $c \in S$ such that $a \wedge b \leq c$. Choose $d \in S$ such that $a,b,$ and $c \in (d]$ and put $L = (d]$. Since $S/\theta L$ is distributive, there exists $e, f \in S$ suth that $\pi_L(a) \leq \pi_L(e)$, $\pi_L(b) \leq \pi_L(f)$ and $\pi_L(e) \wedge \pi_L(f) = \pi_L(c)$ x,y and $z \in F$ such that $e \wedge f \wedge x = c \wedge x$, $e \wedge a \wedge y = a \wedge y$ and $f \wedge b \wedge z = b \wedge z$ which implies that $e \wedge f \wedge x = c$, $e \wedge a = a$ and $f \wedge b = b$. Therefore and $(e \wedge x) \wedge (f \wedge x) = c$. Hence, S is distributive.

Theorem 3.3. Let L be a sub-semilattice of S. If J is an ideal of $S/\theta L$, $I = \{x \in S : \pi_L(x) \in J\}$ is an ideal of S. Then J is a proper ideal of $S/\theta L$ if and only if $I \cap L = \emptyset$.

Also, if P is a prime ideal of S such that $P \cap L = \emptyset$ then $Q = \{\pi_L(x) \in S/\theta L : x \in P\}$ is a prime ideal of $S/\theta L$.

Proof: Let J be an ideal of $S/\theta L$, and $I = \{x \in S : \pi_L(x) \in J\}$, let $a \in I$, $b \in S$ such that $b \leq a$. Then $b \wedge x \leq a$ for some $x \in L \Rightarrow \pi_L(b) = \pi_L(x \wedge b) \leq \pi_L(a) \Rightarrow \pi_L(b) \in J$ and hence $b \in I$. Therefore I is an initial segment of S. If $\pi_L(a)$ and $\pi_L(b) \in J$;

$$\Rightarrow \exists c \in S \text{ such that } \pi_L(c) \in J,\ \pi_L(a) \leq \pi_L(c) \text{ and } \pi_L(b) \leq \pi_L(c)$$

$$\Rightarrow \exists x, y \in L \text{ such that } a \wedge c \wedge x = a \wedge x \text{ and } b \wedge c \wedge y = b \wedge y$$

$$\Rightarrow a \wedge c \wedge x \wedge y = a \wedge x \wedge y \text{ and } b \wedge c \wedge x \wedge y = b \wedge x \wedge y$$

$$\Rightarrow \exists d \in S, \text{ such that } a \leq d,\ b \leq d \text{ and } d \wedge c \wedge x \wedge y = d \wedge x \wedge y$$

$$\Rightarrow \pi_L(d) \leq \pi_L(c) \text{ and hence } d \in I.$$

Therefore J is an ideal. If $a \in I \cap L$, then for any $x \in S$, $\pi_L(x) = \pi_L(x \wedge a)$ thus $S/\theta L = J$.

Conversely if $S/\theta L = J$, then for any $x \in L$, $\pi_L(x) \in J$ and therefore $I \cap L = \emptyset$. Also, let P be a prime ideal of S such that $P \cap L = \emptyset$, then Q is proper ideal of $S/\theta L$ and let $\pi_L(a) \wedge \pi_L(b) \in Q$, $\pi_L(a \wedge b) \in Q$.Then $a \wedge b \in P$ and since P is prime $a \in P$ or $b \in P$.

Hence $\pi_L(a) \in Q$ or $\pi_L(b) \in Q$. Therefore Q is prime.

Theorem 3.4. Let L and M be sub-semilattices of S such that $L \subset M$ and let $f: S/\theta L \to S/\theta M$ if and only if $I \cap L = \emptyset$. be the epimorphism defined by $f(\pi_L(x)) = \pi_M(x)$,

If f is injection, then $P \cap L = \emptyset$ implies $P \cap M = \emptyset$, for any prime ideal P of S.

Proof: Suppose P be a prime ideal S of such that $P \cap L = \emptyset$ then by Theorem 3.2 above, $\{\pi_L(x) \in S/\theta L\}$ is a prime ideal of $S/\theta L$ and hence $\{\pi_M(x) \in S/\theta M : x \in P\}$ is a prime ideal of $S/\theta M$, so that $P \cap M = \emptyset$.

Theorem: 3.5. Let F be a filter of S, then the following are equivalent:

1. F is a prime filter
2. $F = \{\pi_F(x) : x \in S - F\}$ is an ideal of $S/\theta F$
3. $S/\theta F$ has a unique maximal ideal.

Proof: (i) -> (ii) Suppose that F is a prime filter, then $S - F$ is a prime ideal.

Let $\pi_F(x) \in I$ and $\pi_F(y) \in S/\theta F$ such that $\pi_F(y) \leq \pi_F(x)$

$$\Rightarrow \exists a \in F \text{ such that } y \wedge a = x \wedge y \wedge a \Rightarrow y \wedge a \leq x$$

$$\Rightarrow \pi_F(y) = \pi_F(y \wedge a) \leq \pi_F(x) \Rightarrow \pi_F(y) \leq \pi_F(x).$$

Therefore I is an initial segment. Let $\pi_F(x), \pi_F(y) \in I$ this implies $x,y \in S - F$. Then there exist $z \in S - F$ such that $x \leq z$ and $y \leq z$.

Again, there exists $a \in F$ such that $x \wedge a = x \wedge a \wedge z$ and $y \wedge a = y \wedge a \wedge z$

$$\Rightarrow x \wedge a \leq z \text{ and } y \wedge a \leq z$$

$$\Rightarrow \pi_F(x) = \pi_F(x \wedge a) \leq \pi_F(z) \text{ and } \pi_F(y) = \pi_F(y \wedge a) \leq \pi_F(z)$$

$$\Rightarrow \pi_F(x) \leq \pi F(z) \text{ and } \pi_F(y) \leq \pi F(z).$$

Therefore $I = \{\pi_F(x) : x \in S - F\}$ is an ideal of S/θ_F. Hence the result.

(ii) -> (iii): For any $a \in F$, $\pi_F(a)$ is the greatest element of S/θ_F. Hence, every ideal of S/θ_F is contained in $\{\pi_F(x) : x \in S - P\}$.

Therefore S/θ_F has a unique maximal.

(iii)-> (i): Let G be the unique maximal ideal of S/θ_F and let x and $y \in S - F$. Since the principal ideals of S/θ_F generated by $\pi_F(x)$ and $\pi_F(y)$ are proper, and it follows that $\pi_F(x)$ and $\pi_F(y) \in G$. Then, there exist $\pi_F(z) \in G$ such that $\pi_F(x) \leq \pi_F(z)$ and $\pi_F(y) \leq \pi_F(z)$. This implies $x \wedge z \wedge a = x \wedge a$ and $y \wedge z \wedge a = y \wedge a$ for some $a \in F$.

Then, there exist $b \in S$ such that $x \leq b$, $y \leq b$ and $b \wedge z \wedge a = b \wedge a \Rightarrow b \in S - F$, and thus S-F is a prime ideal. Henceforth F is prime filter.

Theorem 3.6. Let F be a filter of S. Then F is a maximal filter if and only if S/θ_F is the two-element chain.

Proof: Suppose F is a maximal filter, then for any x, and $y \in F$, $\pi_F(x) = \pi_F(y)$ and for any $x \in F$ and $y \in S$, $\pi_F(x) = \pi_F(y)$ implies $y \in F$.

Now, let x and $y \in S$-F, Since F is maximal, there exist $a,b \in F$, such that $a \wedge x \leq y$ and $b \wedge y \leq x$. Hence $\pi_F(x) = \pi_F(a \wedge x) \leq \pi_F(y) = \pi_F(b \wedge y) \leq \pi_F(x)$.

Therefore $\pi_F(x) = \pi_F(y)$.

Conversely, suppose that S/θ_F is the two-element chain. Suppose x, and $y \in S$-F. Choose an element $z \in F$, such that $\pi_F(z) \neq \pi_F(x)$. Hence $\pi_F(x) = \pi_F(y)$. Then there exist $a \in F$ such that $x \wedge a = y \wedge a$ and $y \in F \vee [x)$.

Therefore, for any $x \notin F$, $F \vee [x) = S$. Thus F is maximal. Hence the result.

Theorem 3.7. The map $F \longmapsto \theta_F$ is an isomorphism from the lattice $F_0(S)$ of all non-empty filters S of into a permutable sublattice of the lattice $C(S)$ of all congruencies on S. Hence $\{\theta_F : F \in F_0(S)\}$ is a distributive and permutable sub lattice of $C(S)$.

Proof: Suppose I and J are any filters of S. Then $\theta_{I \cap J} \subset \theta_I \cap \theta_J$ and let $(x,y) \in \theta_I \cap \theta_J$, there exist $a \in I$, $b \in J$ such that $x \wedge a = y \wedge a$ and $x \wedge b = y \wedge b$.

$$\Rightarrow c \in S \text{ such that } a \leq c,\ b \leq c \text{ and } x \wedge c = y \wedge c$$

Now since $c \in I \cap J$, then $(x,y) \in \theta_{I \cap J}$. Hence $F \longmapsto \theta_F$ is a lattice homomorphism.

Also let us consider $I, J \in F_0(S)$ and $\theta_I \subset \theta_J$. Let $x \in I$, choose $y \in J$ then there exist $z \in I$ such that $x \leq z$ and $y \leq z$.

Now $(x,y) \in \theta_I \subset \theta_J$ and since $z \in J$, we have $x \in J$. There $I \subset J$ fore and hence $F \longmapsto \theta_F$ is a lattice isomorphism of $F_0(S)$ onto the sub lattice $\{\theta_F : F \in F_0(S)\}$ of $C(S)$.

Note that the lattice $C(S)$ of all congruencies on a semilattice S need not be distributive, even when S is distributive. Hence the result.

Example 3.7. Let $S = \mathbb{Z}^+ \cup \{0, a, b\}$, with partial order given by $0 < a < n$, $0 < b < n$, for all andwhen everis positive for $n \in \mathbb{Z}^+$ and $m<n$ when ever m-n is positive for all $m, n \in \mathbb{Z}^+$.

Then S becomes distributive semilattice, with $a \wedge b = glb\{a,b\}$ $\forall$ $a,b \in S$

CONCLUSION

In general, we introduced the notation of natural epimorphism of a semilattice on the quotient semilattice. Let L and M be sub-semilattices of S such that $L \subset M$ and let $f : S/\theta_L \rightarrow S/\theta_M$ be the epimorphism defined by $f(\pi_L(x)) = \pi_M(x)$. Also, we can conclude that if f is injection map, then $P \cap L = \emptyset$ implies , $P \cap M = \emptyset$ for any prime ideal P of S. Additionally, for a distributive semilattice S, let F be a filter of S, then θ_F is the smallest congruence on S containing F in a single equivalence class and that S/θ_F is distributive. Similarly, map $F \longmapsto \theta_F$ is an isomorphism from the lattice $F_0(S)$ of all non-empty filters of into a permutable sublattice of S the lattice $C(S)$ of all congruencies on S---.

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